NOTES ON NON-ARCHIMEDEAN JULIA SETS

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1. Non-archimedean Julia sets

These are notes for a learning talk to accompany Chapter 8, *Introduction to Dynamics on Berkovich Space*, of Benedetto's *Dynamics in One Non-Archimedean Variable* [Ben19].

1.1. **Comparative Fatou-Julia Theory.**

Definition 1.1. A set of self-maps S of a metric space (X, d) is called *equicontinuous* on a subset $U \subseteq X$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for all maps ϕ in S, we have for all $x, y \in U$ that

$$d(x,y) < \delta \implies d(f(x), f(y)) < \epsilon.$$

This is much more restrictive than asking for all the maps ϕ in S to continuous. For continuity at x, we need to say that we can arrange for $f(x)$ and $f(y)$ to be close together (within ϵ) as long as x and y are sufficiently close together by $\delta = \delta(\epsilon, x)$. For uniform continuity, we require further that for each map $f \in S$, the same closeness $\delta = \delta(\epsilon, f)$ works across the whole domain U, independent of x. Equicontinuity says further that the same uniform continuity bound δ works for every map $f \in S$, independent of f.

Example 1.2. If ϕ is 1-Lipschitz, then all its iterates are also 1-Lipschitz, so we can choose $\delta(\epsilon) = \epsilon$ to prove that the set of iterates ϕ^n is equicontinuous.

Example 1.3. Consider the map ϕ of the real interval $[0,1]$ that sends $x \mapsto x^2$. The set of all iterates of this map is not equicontinuous in the real topology because, near 1, there are points that eventually go near 0. Nevertheless, the iterates are equicontinuous on the forward-invariant subset $[0,1)$. To see this, let $x \in [0,1)$ and consider a ball around x of some small radius. After iterating enough times, that ball will be compressed to within the interval $[0, 1/4)$, where the map ϕ is 1-Lipschitz. The so first few iterates impose some restrictions on δ, but after those finitely many, there are no more restrictions, so we can find δ as the minimum of the uniform continuity constants for the first few iterates.

Definition 1.4 (Complex Fatou)**.** On $\mathbb{P}^1(\mathbb{C})$, the *Fatou set* or *domain of equicontinuity* of a rational map ϕ of degree at least 2 is the set of points admitting a neighborhood on which the family of iterates $(\phi^n)_{n=1}^{\infty}$ is equicontinuous.

An equivalent definition comes from considering when the forward images of a subset almost cover all of $\mathbb{P}^1$. Given a sequence of sets $U_1, U_2, \ldots$, we say that sequence *omits n points* if

$$\#\left(\mathbb{P}^1 \setminus \bigcup_{i=1}^{\infty} U_i\right) \geq n.$$

Note that the sequence may omit more than exactly n points.

Definition 1.5 (Classical Fatou v2)**.** On $\mathbb{P}^1(\mathbb{C})$, the *Fatou set* of a rational map ϕ of degree at least 2 is the set of points admitting a neighborhood U with the property that the sequence of forward images $\phi^n(U)$ omits 3 points.

Definition 1.6 (Classical Fatou v3)**.** On $\mathbb{P}^1(\mathbb{C})$, the *Fatou set* of a rational map ϕ of degree at least 2 is the set of points admitting a neighborhood U with the property that the sequence of forward images $\phi^n(U)$ omits uncountably many points.

There is another formulation of the Fatou set that we will not get into here, in terms of normality; it is equivalent to the other versions by the Arzela-Ascoli theorem, but this theorem will fail in non-archimedean contexts.

The equivalence of Definitions 1.4, and 1.5, and 1.6 is one of the first results of complex dynamics. It uses Montel's theorem.

Definition 1.7 (Non-archimedean Classical Fatou)**.** On $\mathbb{P}^1(\mathbb{C}_v)$, where $\mathbb{C}_v$ is a non-archimedean field, the *Fatou set* of a rational map ϕ of degree at least 2 is the domain of equicontinuity.

Definition 1.8 (Non-archimedean Classical Fatou v2)**.** On $\mathbb{P}^1(\mathbb{C}_v)$, where $\mathbb{C}_v$ is a non-archimedean field, the *Fatou set* of a rational map ϕ of degree at least 2 is the set of points admitting a neighborhood U that omits 2 points.

Definition 1.9 (Non-archimedean Classical Fatou v3)**.** On $\mathbb{P}^1(\mathbb{C}_v)$, where $\mathbb{C}_v$ is a non-archimedean field, the *Fatou set* of a rational map ϕ of degree at least 2 is the set of points admitting a neighborhood U that omits uncountably many points.

These three definitions are all equivalent by Hsia's non-Archimedean Montel theorem, which tells us about maps on disks that omit values.

In Berkovich space, we adapt the omitted-points definition to get a notion of Fatou set.

Definition 1.10. On $\mathbb{P}^1_{\mathrm{an}}$, given a rational map ϕ of degree at least 2, an open set $U \subset \mathbb{P}^1_{\mathrm{an}}$ is called *dynamically stable* if its sequence of iterated images omits infinitely many points.

This is not to be confused with other notions of dynamical stability such as J-stability.

Definition 1.11 (Berkovich Fatou)**.** On $\mathbb{P}^1_{\mathrm{an}}$, the *Fatou set* $\mathcal{F}_{\mathrm{an}}$ of a rational map ϕ of degree at least 2 is the set of points admitting a dynamically stable neighborhood.

There are connections from this definition to equicontinuity and normality, but this is a complicated story; see [FKT12].

Remark 1.12. Over $\mathbb{C}$, we had the principle that if the forward images of U omit at least 3 points, they in fact omit uncountably many points. This makes one wonder if one can check dynamical stability by finding finitely many omitted points, which would give an easier-to-check condition for being in the Berkovich Fatou set. Indeed, with one class of exceptions, if the iterates of a set U by ϕ omit 3 values, then U is dynamically stable. The exceptions are exactly $\mathbb{C}_v = \mathbb{C}_p$ and $\phi(z) = z^{p^m}$ for some prime p and integer $m \geq 1$. This is [Ben19, Exercise 8.1].

Definition 1.13 (Julia)**.** In each of these contexts, the Julia set is defined as the complement of the Fatou set:

$$\mathcal{J} := \mathbb{P}^1(\mathbb{C}) \smallsetminus \mathcal{F}, \quad \mathcal{J}_{\mathrm{I}} := \mathbb{P}^1(\mathbb{C}_v) \smallsetminus \mathcal{F}_{\mathrm{I}}, \quad \mathcal{J}_{\mathrm{an}} := \mathbb{P}^1_{\mathrm{an}} \smallsetminus \mathcal{F}_{\mathrm{an}}.$$

The Berkovich Fatou-Julia theory extends the classical one, in the sense that the classical parts of the Berkovich Fatou and Berkovich Julia sets show up as the classical Fatou and Julia sets.

Theorem 1.14. *For any rational map ϕ of degree at least 2 on $\mathbb{C}_v$, we have*

$$\mathcal{F}_{\mathrm{I}} = \mathcal{F}_{\mathrm{an}} \cap \mathbb{P}^1(\mathbb{C}_v), \quad \mathcal{J}_{\mathrm{I}} = \mathcal{J}_{\mathrm{an}} \cap \mathbb{P}^1(\mathbb{C}_v).$$

For polynomials, one can also consider the filled Julia set over $\mathbb{C}$, which is somewhat easier to visualize since it has an interior.

Definition 1.15 (Filled Julia)**.** In each of these context, the *filled Julia set* of a polynomial of degree 2 is defined as the set of points with bounded orbit:

$$\mathcal{K} := \{p \in \mathbb{C} : \lim_{n\to\infty} \phi^n(p) \neq \infty\},$$
$$\mathcal{K}_{\mathrm{I}} := \{p \in \mathbb{C}_v : \lim_{n\to\infty} \phi^n(p) \neq \infty\},$$
$$\mathcal{K}_{\mathrm{an}} := \{p \in \mathbb{A}^1_{\mathrm{an}} : \lim_{n\to\infty} \phi^n(p) \neq \infty\}.$$

Remark 1.16. For the Berkovich filled Julia set, we should think about what it means for a sequence of Berkovich points ζ_n to converge to ∞ in the Gel'fand topology. Let's just assume all the ζ_n have the same type, since this is what comes up for our application. For Type I points, convergence to ∞ just means that the absolute values of the points go to ∞. For Type II, III, and IV points, the ζ_n converge to ∞ if, viewed as affine disks, the diameters go to ∞.

Remark 1.17 (Setting expectations)**.** All these Fatou sets are open, because they are defined by open conditions. Hence all these Julia sets are closed.

- Over $\mathbb{C}$, the Julia set is a closed, perfect, nonempty, compact set. Its interior is empty or all of $\mathbb{P}^1$, which happens e.g. for Lattès maps. Its interior is empty, except for the Lattès example, which has Julia set all of $\mathbb{P}^1$. (Recall that a perfect set is a closed set with no isolated points). The filled Julia set is closed, compact, and bounded, and the Julia set is its boundary:

$$\mathcal{J} = \partial\mathcal{K}.$$

- Over $\mathbb{P}^1(\mathbb{C}_v)$, the Julia set is closed, perfect, and has empty interior. But it can be empty. Indeed it is empty for many examples of interest, including maps of good reduction. The filled Julia set is closed and bounded, but not necessarily compact. (The familiar argument that a closed and bounded subset of a locally compact metric space is compact fails here – no local compactness.) The Julia set is its boundary:

$$\mathcal{J}_{\mathrm{I}} = \partial\mathcal{K}_{\mathrm{I}}.$$

  But the topology is non-intuitive. For instance, the map $z \mapsto z^2$ has filled Julia set $\bar{D}(0,1)$; the boundary is empty, and $\mathcal{J}_{\mathrm{I}} = \varnothing$.
- The Berkovich Julia set is closed, nonempty, compact, and has empty interior. It is either a singleton or perfect, and both these possibilities do occur. The filled Berkovich Julia set is closed, compact, and bounded, and its boundary is the Berkovich Julia set:

$$\mathcal{J}_{\mathrm{an}} = \partial\mathcal{K}_{\mathrm{an}}.$$

The following theorem gives us a supply of Berkovich Julia points.

Theorem 1.18. *For any rational map ϕ of degree at least 2, every repelling Type II periodic point is in the Berkovich Julia set $\mathcal{J}_{\mathrm{an}}$.*

The Berkovich Julia set detects good reduction.

Theorem 1.19. *Let ϕ be a rational map of degree at least 2. The following are equivalent:*

1. *The map ϕ has explicit good reduction.*
2. *The Gauss point is a repelling fixed point of degree d.*
3. *The Gauss point is a totally ramified fixed point.*
4. *The Berkovich Julia set is $\mathcal{J}_{\mathrm{an}} = \{\zeta_{\mathrm{Gauss}}\}$.*

Since we can change coordinates to move any Type II point to ζ_{Gauss}, the map ϕ has potential good reduction if and only if $\mathcal{J}_{\mathrm{an}}$ is a single Type II point.

Instead of proving the theorem, we will illustrate it in the case of $z^2 + \lambda z$ where $|\lambda| \le 1$, below.

Remark 1.20 (Why this?)**.** The real reason that this definition of the Berkovich Julia set is the right analogue of the Julia set to consider is that it admits an equilibrium measure, so it can be studied using ergodic theory. But even without going there, we can see some evidence that the Berkovich Julia set is better than the classical non-archimedean Julia set.

- It is nonempty.
- It detects good reduction.

A key ingredient is the following lemma.

Lemma 1.21 (Branch repulsion lemma)**.** *Suppose that ϕ is a rational map that sends $\vec{0}$ to $\vec{0}$ at ζ_{Gauss} with local degree at least 2 in that direction. In particular ϕ fixes the Gauss point. Then every Berkovich neighborhood of ζ_{Gauss} omits at most 1 point in $\vec{0}$ upon iteration.*

For the proof, see Benedetto. A good example to keep in mind is $z \mapsto z^2$, which can't hit $0 \in \mathbb{P}^1(\mathbb{C}_v)$ on the nose.

1.2. **Examples.** Let K be an algebraically closed, complete non-archimedean field (as usual in these notes). Every quadratic polynomial over K is conjugate to one of the form $z \mapsto z^2 + \lambda z$, because a change of coordinates can be used to move a fixed point to 0. The two cases $|\lambda| \le 1$ and $|\lambda| > 1$ have very different non-archimedean dynamics.

Example 1.22. Let $\phi(z) = z^2 + \lambda z$, where $|\lambda| \le 1$. The reduction of ϕ modulo the maximal ideal of K is simply $\bar{\phi}(z) = z^2$. The degree did not drop, so ϕ has good reduction.

We claim that

$$\mathcal{J}_{\mathrm{an}} = \{\zeta_{\mathrm{Gauss}}\}.$$

Let us check directly that $\mathcal{J}_{\mathrm{I}} = \varnothing$. The map ϕ is 1-Lipschitz on sufficiently small $\mathbb{P}^1$-disks, since it has good reduction. Now given any point in $\mathbb{P}^1(\mathbb{C}_v)$, surround it by a small $\mathbb{P}^1$-disk; this is a classical neighborhood witnessing equicontinuity.

Separately from the above argument, we can show that $\mathcal{J}_{\mathrm{an}} \subseteq \{\zeta_{\mathrm{Gauss}}\}$. Let ζ be any Berkovich point besides ζ_{Gauss}. Then we may surround ζ by a Berkovich affinoid U that is contained within one direction from ζ_{Gauss}. Since ϕ has good reduction, it maps directions at

ζ_{Gauss} to directions at ζ_{Gauss}; or equivalently, it maps residue classes to residue classes. Let v be the direction containing U.

If the residue field k of K is countable, then k is the algebraic closure of a finite field, and the orbit of v in $\mathbb{P}^1(k)$ is necessarily finite. Then $\bigcup_{n=1}^{\infty} \phi^n(U) \subset \bigcup_{n=1}^{\infty} \phi^n(v)$ omits infinitely many directions at ζ_{Gauss}, and these each contain many points, so U is a dynamically stable neighborhood.

If the residue field k of K is uncountable, then the same argument works, except now we use the fact that the uncountable set of directions can't all be hit with the countably many iterates $\phi^n(v)$.

To see that $\zeta_{\text{Gauss}} \in \mathcal{J}_{\text{an}}$ is more difficult. We must show that every neighborhood U of ζ_{Gauss} omits only finitely many points upon iteration. To see this, let $V \subseteq U$ be an open Berkovich affinoid containing ζ_{Gauss}, which exists by the construction of the Gel'fand topology. Then there are only finitely many directions $v_1, \ldots, v_N$ out from ζ_{Gauss} that are not completely contained in V, by the definition of affinoid. Hence the set of omitted points of $\bigcup_{n=1}^{\infty} \phi^n(U)$ is contained in a subset of that set of directions, which is necessarily a finite set $\{v_1, \ldots, v_n\}$. This set is clearly backwards invariant for ϕ, and finite, hence totally invariant and all orbits are periodic.

Now, replacing ϕ with an iterate ϕ^m cannot cause the number of omitted points of the set U to go down, so it suffices to show that for some iterate ϕ^m, the set of omitted points is finite. Let ϕ^m be an iterate that maps each of the directions $v_1, \ldots, v_n$ to itself; this exists because each of these directions is periodic for the action of ϕ on the reduction $\mathbb{P}^1(k)$. From now on we write ϕ in place of ϕ^m.

Let us show that v_1 contains at most one omitted point for the iterates of U. Changing coordinates, we may assume that v_1 is the direction $\vec{0}$ of 0 at ζ_{Gauss}. Since $\phi^{-1}(\vec{0}) = \{\vec{0}\}$, the local degree of ϕ along $\vec{0}$ is $\deg \phi = 2$. Because U is an affinoid, it contains some Berkovich annulus centered at 0 with outer radius 1. Then use Lemma 1.21.

Example 1.23. Consider the following quadratic polynomial with bad reduction:

$$\phi(z) = z^2 + \lambda z,$$

where $|\lambda| > 1$.

The Type I Julia set can be described constructively as a Cantor set. To describe it, we start with the disk $U_0 = \bar{D}(0, |\lambda|)$. Any point $z \in \mathbb{P}^1(\mathbb{C}_v)$ that is not in U_0 is in $\mathcal{F}_{\text{I}}$, since

$$|z| > |\lambda| \implies |\phi(z)| = |z|^2$$

by the ultrametric triangle inequality. Upon iteration, we see that $|z| \to \infty$.

This implies that all iterated inverse images of U_0 are also in $\mathcal{F}_{\text{I}}$. Let these be called $U_1, U_2, \ldots$. Their intersection is a bounded, forward-invariant set, so their intersection is contained in the filled Julia set $\mathcal{K}_{\text{I}}$. In fact $\mathcal{K}_{\text{I}}$ is exactly this nested intersection, since any point not in the nested intersection has unbounded orbit by the argument just given. So the filled Julia set is this Cantor set. The boundary of a Cantor set is itself, so $\mathcal{J}_{\text{I}} = \mathcal{K}_{\text{I}}$.

We claim that $\mathcal{J}_{\text{I}} = \mathcal{J}_{\text{an}}$. To see this, first let ζ be a Type II or III point of some diameter r. Eventually the finitely-many constituent disks of U_i are smaller in diameter than r, so for some i, the set $\zeta \smallsetminus U_i$ is nonempty. Thus $\phi^n(\zeta)$ includes a point escaping to ∞, so $\zeta \to \infty$.

If $\zeta = \lim \zeta_n$ is a Type IV point, where ζ_n is a decreasing sequence of disks all with diameter greater than r, then again there exists U_i such that, for sufficiently large n, all the sets $\zeta_n \smallsetminus U_i$

are empty. Then all these disks $\phi^i(\zeta_n)$ are close to ∞, and $\phi^i(\zeta)$ is their limit, so $\phi^i(\zeta)$ is close to infinity.

Example 1.24 (Lattés with bad reduction)**.** We now consider the Lattés map

$$\phi(z) = \frac{z^4 - 8c^4 z^2 - c^4}{4z^3 + z^2 + 4c^4}. \tag{1}$$

If $|c|$ is at least 1, this map has good reduction and so $\mathcal{J}_{\mathrm{an}} = \{\zeta_{\mathrm{Gauss}}\}$.

Suppose now that $0 < |c| < 1$. We claim that the Julia set is a Berkovich interval,

$$\mathcal{J}_{\mathrm{an}} = [\zeta(0, |c|^2), \zeta(0, |c|)].$$

We prove this directly from the formula for $\phi(z)$, without using any properties of Lattés maps.

First we show that $\mathcal{J}_{\mathrm{I}} = \varnothing$. The proof method for this is totally explicit. Using Newton polygons, one can show the following behavior for Type I points $z \in \mathbb{P}^1(\mathbb{C}_v)$ according to the absolute value $|z|$.

(1) If $|z|$ is in $(1, \infty)$, then $|\phi(z)| = |\phi(z)|^2$. This comes from computing the absolute value of the dominant term in the numerator and denominator in (1). So all such z are in $\mathcal{F}_{\mathrm{I}}$.
(2) If $|z| = 1$, then either $|\phi(z)| > 1$, in which case $z \in \mathcal{F}_{\mathrm{I}}$ due to the previous bullet; or the disk $D(z, 1)$ is mapped to the disk $D(\phi(z), 1)$ where $|\phi(z)| = 1$. If all iterates of z have the latter behavior, the iterates of ϕ are 1-Lipschitz near z, so either way $z \in \mathcal{F}_{\mathrm{I}}$.
(3) If $|z| \le |c|^2$, then either $\phi(z) > 1$, so by the first bullet we have $z \in \mathcal{F}_{\mathrm{I}}$, or $\phi(D(x, |c|^2) \subseteq D(\phi(x), 1)$. In the latter case, the second bullet tells us that $z \in \mathcal{F}_{\mathrm{I}}$.
(4) If $|z|$ is in the range $(|c|^2, |c|)$, then $|\phi(z)| < |c|^2$, and so by the third bullet we have $z \in \mathcal{F}_{\mathrm{I}}$.
(5) We have explained the behavior for all z except possibly $|z| = |c|$. In this case we have $|\phi(x)| \le |c|^2$, so the third bullet tells us that $z \in \mathcal{F}_{\mathrm{I}}$.

Notice the method of proof here was to partition $\mathbb{P}^1(\mathbb{C}_v)$ into several "intervals" which were mapped to each other in an understandable way. It was easier to understand some of these properties in terms of disks, which points us naturally to Berkovich space.

We still have a nonempty Berkovich Julia set. Indeed, the Gauss point is a repelling fixed point of degree 2, since $\deg \bar{\phi} = 2$. Thus $\zeta_{\mathrm{Gauss}} \in \mathcal{J}_{\mathrm{an}}$, and all iterated preimages of ζ_{Gauss} are in $\mathcal{J}_{\mathrm{an}}$. These iterated images turn out to be dense in the interval $I := [\zeta(0, |c|^2), \zeta(0, 1)]$, and the Julia set is closed, so the whole interval is in the Julia set. We see this by describing the dynamics on I exactly. Let $I_1 = [\zeta(0, |c|^2), \zeta(0, |c|)]$ and let $I_2 = [\zeta(0, |c|), \zeta(0, 1)]$. Then ϕ scales I_2 to $I_1 \cup I_2$ and I_1 to $I_1 \cup I_2$ in the reverse direction. This means we have "tent dynamics."

The last step is to show that no points outside I are in $\mathcal{J}_{\mathrm{an}}$. The branch $\vec{\infty}$ at ζ_{Gauss} is mapped to itself, so that branch is all in Fatou (the direction $\vec{\infty}$ gives us a dynamically stable neighborhood of points in $\vec{\infty}$). If a disk has center z with $|z| = 1$, the second bullet above controls the size of the image disks, etc.

References

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